Packet I – $\boxed{\text{KEY}}$

Find the power series representation for $f(x)$ and specify the radius of convergence. Each is somehow related to a geometric series.

1. $f(x) = \dfrac{1}{1+x}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

Let $x \to -x$

$$\frac{1}{1-(-x)} = 1 + (-x) + (-x)^2 + (-x)^3 + \ldots$$
$$= 1 - x + x^2 - x^3 + \ldots$$
$$= \sum_{n=0}^{\infty} (-1)^n x^n$$

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(-1)^n x^n} \right| < 1$$

$|x| < 1$

$(-1, 1)$ – interval of conv.

radius of conv. $= 1$

2. $f(x) = \dfrac{1}{(1+x)^2}$

Hint: Differentiate problem 1.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

$$D_x\left(\frac{1}{1+x}\right) = D_x\left(1 - x + x^2 - \ldots\right)$$

$$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + \ldots$$

$$\therefore \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \ldots$$
$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n$$

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} (n+2) x^{n+1}}{(-1)^n (n+1) x^n} \right| < 1$$

$|1 \cdot x| < 1$

$|x| < 1$

$(-1, 1)$

radius $= 1$

3. $f(x)=\dfrac{1}{(1-x)^3}$

$$-\frac{1}{(1+x)^2} = -1 + 2x - 3x^2 + \dots$$

$$D_x\left(-\frac{1}{(1+x)^2}\right) = D_x\left(-1+2x-3x^2+\dots\right)$$

$$\frac{2}{(1+x)^3} = 2 - 6x + 12x^2 - \dots$$

Let $x \to -x$

$$\frac{2}{(1+(-x))^3} = 2 - 6(-x) + 12(-x)^2 - \dots$$

$$\frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + \dots$$

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + \dots$$

Notice 1, 3, 6, ... are triangular numbers following the equation $\frac{n(n+1)}{2}$

$$\therefore = \sum_{n=1}^{\infty} \frac{n(n+1)}{2} x^{n-1}$$

$$\rho = \lim_{n\to\infty}\left|\frac{\frac{(n+1)(n+2)}{2}x^n}{\frac{n(n+1)}{2}x^{n-1}}\right| < 1$$

$|x| < 1$

$(-1, 1)$

radius $= 1$

4. $f(x)=\dfrac{x}{(1+x)^2}$

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - \dots$$

$$\frac{x}{(1+x)^2} = x - 2x^2 + 3x^3 - \dots$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^{n+1}$$

$$\rho = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+2)x^{n+2}}{(-1)^n(n+1)x^{n+1}}\right| < 1$$

$|x| < 1$

$(-1, 1)$

radius $= 1$

5. $f(x) = \frac{1}{2-3x} = \frac{\frac{1}{2}}{1-\frac{3}{2}x}$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

Let $x \to \frac{3}{2}x$

$$\frac{1}{1-\frac{3}{2}x} = 1 + \frac{3}{2}x + \left(\frac{3}{2}x\right)^2 + \ldots$$

$$\frac{1/2}{1-3/2x} = \frac{1}{2} + \frac{3}{4}x + \frac{9}{8}x^2 + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{3^n}{2^{n+1}} x^n$$

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{3^{n+1}}{2^{n+2}} x^{n+1}}{\frac{3^n}{2^{n+1}} x^n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{3^{n+1}}{3^n} \cdot \frac{2^{n+1}}{2^{n+2}} \cdot x \right| < 1$$

$$\left| \frac{3}{2} x \right| < 1$$

$$|x| < \frac{2}{3}$$

$$\left(-\frac{2}{3}, \frac{2}{3}\right)$$

radius $= \frac{2}{3}$

6. $f(x) = \frac{1}{3+2x}$

$$\frac{1}{3+2x} = \frac{1/3}{1+2/3x}$$

$$= \frac{1}{3} \cdot \frac{1}{1+2/3x}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots$$

Let $x \to \frac{2}{3}x$

$$\frac{1}{3} \cdot \frac{1}{1+2/3x} = \frac{1}{3} \cdot \left(1 - \frac{2}{3}x + \left(\frac{2}{3}x\right)^2 - \ldots\right)$$

$$= \frac{1}{3}\left(1 - \frac{2}{3}x + \frac{4}{9}x^2 - \frac{8}{27}x^3 + \ldots\right)$$

$$= \frac{1}{3} - \frac{2}{9}x + \frac{4}{27}x^2 - \frac{8}{81}x^3 + \ldots$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^n}{3^{n+1}} x^n$$

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \frac{2^{n+1}}{3^{n+2}} x^{n+1}}{(-1)^n \frac{2^n}{3^{n+1}} x^n} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{2^{n+1}}{2^n} \cdot \frac{3^{n+1}}{3^{n+2}} \cdot x \right| < 1$$

$$\left| \frac{2}{3} x \right| < 1$$

$$|x| < \frac{3}{2}$$

$$\left(-\frac{3}{2}, \frac{3}{2}\right)$$

radius $= 3/2$

7. $f(x) = \dfrac{x^2}{1-x^4}$

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots$$

Let $x \to x^4$

$$x^2\left(\frac{1}{1-x^4}\right) = x^2\left(1 + x^4 + (x^4)^2 + \ldots\right)$$
$$= x^2\left(1 + x^4 + x^8 + \ldots\right)$$
$$= x^2 + x^6 + x^{10} + \ldots$$
$$= \sum_{n=0}^{\infty} x^{4n+2}$$

$$\rho = \lim_{n\to\infty}\left|\frac{x^{4(n+1)+2}}{x^{4n+2}}\right| < 1$$

$$|x^4| < 1$$

$$-1 < x^4 < 1$$

$$-1 < x < 1$$

$(-1, 1)$

radius $= 1$

8. $f(x) = \dfrac{x^3}{2-x^3}$

$$\frac{x^3}{2-x^3} = \frac{\frac{1}{2}x^3}{1-\frac{1}{2}x^3}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots$$

Let $x \to \frac{1}{2}x^3$

$$\frac{\frac{1}{2}x^3}{1-\frac{1}{2}x^3} = \frac{1}{2}x^3\left(1 + \frac{1}{2}x^3 + \left(\frac{1}{2}x^3\right)^2 + \ldots\right)$$
$$= \frac{1}{2}x^3\left(1 + \frac{1}{2}x^3 + \frac{1}{4}x^6 + \ldots\right)$$
$$= \frac{1}{2}x^3 + \frac{1}{4}x^6 + \frac{1}{8}x^9 + \ldots$$
$$= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n x^{3n}$$

$$\rho = \lim_{n\to\infty}\left|\frac{\left(\frac{1}{2}\right)^{n+1} x^{3(n+1)}}{\left(\frac{1}{2}\right)^n x^{3n}}\right| < 1$$

$$\left|\frac{x^3}{2}\right| < 1$$

$$|x^3| < 2$$

$$-2 < x^3 < 2$$

$(-2^{1/3}, 2^{1/3})$: radius $= 2^{1/3}$