"Packet H"

Find the convergence set of the given power series. *Hint*: First find a formula for the *n*th term, then use the Absolute Ratio Test.

1. $\frac{x}{1\cdot 2}-\frac{x^2}{2\cdot 3}+\frac{x^3}{3\cdot 4}-\frac{x^4}{4\cdot 5}+\frac{x^5}{5\cdot 6}-\cdots$

$$\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)(n+2)}(-1)^n$$

$$\rho = \lim_{n\to\infty}\left|\frac{x^{n+2}/(n+2)(n+3)}{x^{n+1}/(n+1)(n+2)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x^{n+2}}{x^{n+1}}\cdot\frac{(n+1)}{(n+3)}\right|$$

$$= \lim_{n\to\infty}|x|\left|\frac{n+1}{n+3}\right| = |x|<1$$

Let $x=-1$

$$\sum_{n=0}^{\infty}(-1)^n(-1)^{n+1}\frac{1}{(n+1)(n+2)}$$

$$=\sum_{n=0}^{\infty}(-1)^{2n+1}\frac{1}{(n+1)(n+2)}$$

$$=\sum_{n=0}^{\infty}-\frac{1}{(n+1)(n+2)} \qquad b_n=\frac{1}{n^2}$$

$\frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$ Converges by DCT

Let $x=1$

$\sum_{n=0}^{\infty}(-1)^n\frac{1}{(n+1)(n+2)}$ converges by AST

$\frac{1}{(n+1)(n+2)} < \frac{1}{n^2}$ converges absolutely

$\therefore\ [-1,1]$

2. $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots$

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$$\rho=\lim_{n\to\infty}\left|\frac{x^{2(n+1)+1}/(2(n+1)+1)!}{x^{2n+1}/(2n+1)!}\right|$$

$$=\lim_{n\to\infty}|x^2|\left|\frac{1}{(2n+2)(2n+3)}\right| = 0<1$$

$\therefore\ (-\infty,\infty)$

(converges for all real numbers.)

3. $x+2x^2+3x^3+4x^4+\cdots$

$$\sum_{n=0}^{\infty} (n+1)\, x^{n+1}$$

$$\rho = \lim_{n\to\infty} \left| \frac{(n+2)\, x^{n+2}}{(n+1)\, x^{n+1}} \right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{n+2}{n+1}\right| = |x|$$

$|x| < 1$

Let $x = 1$

$$\sum_{n=0}^{\infty} (n+1)$$

Diverges by n-th term test

Let $x = -1$

$$\sum_{n=0}^{\infty} (n+1)(-1)^{n+1}$$

$$\lim_{n\to\infty} (n+1)(-1)^{n+1} = \pm\infty$$

Diverges by $n^{th}$ term test

$\therefore\ (-1, 1)$

4. $1-x+\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}-\cdots$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n}$$

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+2}/n+2}{x^{n+1}/n+1} \right|$$

$$= \lim_{n\to\infty} \left| x \cdot \frac{n+1}{n+2} \right|$$

$$= |x| \lim_{n\to\infty} \left|\frac{n+1}{n+2}\right| = |x|$$

$|x| < 1$

Let $x = 1$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n}$$

Converges by A.S.T. (conditionally)

Let $x = -1$

$$\sum_{n=0}^{\infty} (-1)^n \cdot \frac{(-1)^n}{n}$$

$$\sum_{n=0}^{\infty} \frac{1}{n}$$

Diverges by p-series test

$\therefore\ (-1, 1]$

5. $1-\frac{x}{1\cdot 3}+\frac{x^2}{2\cdot 4}-\frac{x^3}{3\cdot 5}+\frac{x^4}{4\cdot 6}-\cdots$

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{n(n+2)} + 1$$

(because $n=0$ is undefined)

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}/(n+1)(n+3)}{x^n/n(n+2)} \right|$$

$$= \lim_{n\to\infty} \left| x \cdot \frac{n(n+2)}{(n+1)(n+3)} \right|$$

$$= |x| < 1$$

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Let $x = -1$

$$\sum_{n=1}^{\infty} (-1)^n \frac{(-1)^n}{n(n+2)} = \sum \frac{1}{n(n+2)}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^2+2n}$$

By direct comparison

$\frac{1}{n^2+2n} < \frac{1}{n^2}$ therefore

$\Sigma$ converges

Let $x = 1$

$\sum_{n=1}^{\infty} (-1) \frac{1}{n(n+2)}$ converges A.S.T.

$\therefore$ $[-1, 1]$

6. $1-\frac{x}{2^1}+\frac{x^2}{2^2}-\frac{x^3}{2^3}+\frac{x^4}{2^4}-\cdots$

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^n}{2^n}$$

$$\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\, x^{n+1}/2^{n+1}}{(-1)^n\, x^n/2^n} \right|$$

$$= \lim_{n\to\infty} \left| x \cdot \frac{1}{2} \right| = \frac{1}{2}|x|$$

$\frac{1}{2}|x| < 1 \Rightarrow |x| < 2$

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Let $x = -2$

$$\sum_{n=0}^{\infty} (-1)^n \frac{(-2)^n}{2^n}$$

$= \sum_{n=0}^{\infty} \frac{2^n}{2^n} = \Sigma 1 \rightarrow$ diverges

Let $x = 2$

$\sum_{n=0}^{\infty} (-1)^n \left(\frac{2^n}{2^n}\right) \rightarrow$ diverges

by $n^{th}$ term test

$\therefore$ $(-2, 2)$

7. $1+2x+\frac{2^2x^2}{2!}+\frac{2^3x^3}{3!}+\frac{2^4x^4}{4!}+\cdots$

$$\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$$

$$\rho = \lim_{n\to\infty} \left| \frac{2^{n+1} x^{n+1} / (n+1)!}{2^n x^n / n!} \right|$$

$$= \lim_{n\to\infty} \left| 2x \cdot \frac{1}{n+1} \right|$$

$$= 2|x| \cdot 0 = 0$$

$$\therefore (-\infty, \infty)$$

8. $\frac{(x-1)^1}{1}+\frac{(x-1)^2}{2}+\frac{(x-1)^3}{3}+\frac{(x-1)^4}{4}+\cdots$

$$\sum_{n=1}^{\infty} \frac{(x-1)^n}{n}$$

$$\rho = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1} / (n+1)}{(x-1)^n / n} \right|$$

$$= \lim_{n\to\infty} \left| (x-1) \cdot \frac{n}{n+1} \right|$$

$$= |x-1| < 1 \Rightarrow (0, 2)$$

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Let $x = 0$

$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n} \rightarrow$ converges by A.S.T.

Let $x = 2$

$\sum_{n=1}^{\infty} (1)^n \frac{1}{n} \rightarrow$ diverges by p-series test

$$\therefore [0, 2)$$

9. $1+\frac{(x+1)^1}{2^1}+\frac{(x+1)^2}{2^2}+\frac{(x+1)^3}{2^3}+\cdots$

$$\sum_{n=0}^{\infty} \frac{(x+1)^n}{2^n}$$

$$\rho = \lim_{n\to\infty} \left| \frac{(x+1)^{n+1}/2^{n+1}}{(x+1)^n/2^n} \right|$$

$$= \lim_{n\to\infty} \left| (x+1)\cdot \frac{1}{2} \right|$$

$$= \frac{|x+1|}{2} < 1$$

$$|x+1| < 2 \Rightarrow (-3, 1)$$

Let $x = -3$

$$\sum_{n=0}^{\infty} \frac{(-2)^n}{2^n} = \sum (-1)^n$$

diverges by $n^{th}$ term test

Let $x = 1$

$$\sum_{n=0}^{\infty} \frac{2^n}{2^n} = \sum 1 \quad \text{diverges}$$

$\therefore (-3, 1)$

10. $\frac{(x+5)^1}{1\cdot 2}+\frac{(x+5)^2}{2\cdot 3}+\frac{(x+5)^3}{3\cdot 4}+\frac{(x+5)^4}{4\cdot 5}+\cdots$

$$\sum_{n=1}^{\infty} \frac{(x+5)^n}{n(n+1)}$$

$$\rho = \lim_{n\to\infty} \left| \frac{(x+5)^{n+1}/(n+1)(n+2)}{(x+5)^n/n(n+1)} \right|$$

$$= \lim_{n\to\infty} \left| (x+5)\cdot \frac{n}{n+2} \right|$$

$$= |x+5| < 1 \Rightarrow (-6, -4)$$

Let $x = -6$

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{n(n+1)}$$

converges by A.S.T.

Let $x = -4$

$$\sum_{n=0}^{\infty} \frac{1}{n(n+1)} = \sum \frac{1}{n^2+n}$$

$\frac{1}{n^2+n} < \frac{1}{n^2}$ converges D.C.T.

$\therefore [-6, -4]$