# Packet G

Show that each of the alternating series converges and then estimate the error made by using the partial sum $S_9$ as an approximation to the sum $S$ of the series.

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{3n+1}$ $\qquad \frac{2}{3n+1} > \frac{2}{3(n+1)+1}$

$\lim_{n\to\infty} \frac{2}{3n+1} = 0$

$\therefore$ By A.S.T., converges

$|R_9| \le \left|\frac{2}{3(10)+1}\right| = \frac{2}{31}$

2. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ $\qquad \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$

$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\therefore$ Converges by A.S.T.

$|R_9| \le \left|\frac{1}{\sqrt{10}}\right| = \frac{1}{\sqrt{10}}$

3. $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1}{\ln(n+1)} > \frac{1}{\ln(n+2)}$

$\lim_{n\to\infty} \frac{1}{\ln(n+1)} = 0$

$\therefore$ By A.S.T., converges

$|R_9| \le \left|\frac{1}{\ln(10+1)}\right| = \frac{1}{\ln 11}$

4. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n} > \frac{\ln(n+1)}{n+1}$

$\lim_{n\to\infty} \frac{\ln n}{n} \overset{L}{=} \frac{1/n}{1} = 0$

$\therefore$ Converges by A.S.T.

$|R_9| \le \left|\frac{\ln 10}{10}\right| = \frac{\ln 10}{10}$

Show that each series converges absolutely.

5. $\sum_{n=1}^{\infty}\left(-\frac{3}{4}\right)^n = \sum_{n=1}^{\infty} (-1)^{n+1}\left(\frac{3}{4}\right)^n$

$\lim_{n\to\infty} \left(\frac{3}{4}\right)^n = 0$

Converges by A.S.T.

$\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$ is a geometric series with $r<1$, converges

$\therefore$ Series converges absolutely.

6. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n\sqrt{n}} > \frac{1}{(n+1)\sqrt{n+1}}$

$\lim_{n\to\infty} \frac{1}{n\sqrt{n}} = 0$

Converges by A.S.T.

$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$

converges by p-series test ($p>1$)

$\therefore$ Series converges absolutely.

7. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n}{2^n} > \frac{n+1}{2^{n+1}}$ $(n>2)$

$\lim_{n\to\infty} \frac{n}{2^n} \overset{L}{=} \lim_{n\to\infty} \frac{1}{\ln 2 \cdot 2^n} = 0$

converges by A.S.T.

$\sum_{n=1}^{\infty} \frac{n}{2^n}$

$\rho = \lim_{n\to\infty} \frac{(n+1)/2^{n+1}}{n/2^n} = \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{2^n}{2^{n+1}}$

$\rho = 1 \cdot \frac{1}{2} = \frac{1}{2} < 1 \Rightarrow$ converges by ratio test

$\therefore$ Series converges absolutely.

8. $\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n^2}{e^n} > \frac{(n+1)^2}{e^{n+1}}$ $(n>2)$

$\lim_{n\to\infty} \frac{n^2}{e^n} \overset{L}{=} \lim_{n\to\infty} \frac{2n}{e^n} \overset{L}{=} \lim_{n\to\infty} \frac{2}{e^n}$

$= 0$

converges by A.S.T.

$\sum_{n=1}^{\infty} \frac{n^2}{e^n}$

$\rho = \lim_{n\to\infty} \frac{(n+1)^2/e^{n+1}}{n^2/e^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \frac{e^n}{e^{n+1}}$

$= 1 \cdot \frac{1}{e} = 1/e < 1 \Rightarrow$ converges by ratio test

$\therefore$ Series converges absolutely.

Classify the series as absolutely convergent, conditionally convergent, or divergent.

9. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{5n}$  $\frac{1}{5n} > \frac{1}{5(n+1)}$

$\lim_{n\to\infty} \frac{1}{5n} = 0 \Rightarrow$ converges by AST

$\sum_{n=1}^{\infty} \frac{1}{5n}$

$\frac{1}{5n} = \frac{1}{5}\cdot\frac{1}{n} \Rightarrow$ diverges

$\therefore$ Series converges conditionally.

10. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{10n+1}$ $> \frac{n+1}{10(n+1)+1}$

$\lim_{n\to\infty} \frac{n}{10n+1} = \frac{1}{10} \neq 0$

AST does not work!

$\lim_{n\to\infty} (-1)^{n+1}\left(\frac{n}{10n+1}\right)$

$= \lim_{n\to\infty} (-1)^{n+1} \cdot \lim_{n\to\infty} \left(\frac{n}{10n+1}\right)$

$= \lim_{n\to\infty} (-1)^{n+1}\cdot 1 = \text{DNE}$

$\therefore$ Diverges by $n^{th}$ Term Test.

11. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$ $> \frac{1}{(n+1)\ln(n+1)}$

$\lim_{n\to\infty} \frac{1}{n\ln n} = 0 \Rightarrow$ converges A.S.T.

$\sum_{n=1}^{\infty} \frac{1}{n\ln n}$

$\int_1^{\infty} \frac{1}{x\ln x}\,dx = \lim_{b\to\infty}\int_1^b \frac{dx}{x\ln x}$

$= \lim_{b\to\infty} \ln(\ln x)\Big|_1^b = \infty$

diverges

$\therefore$ Series converges conditionally.

12. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^4}{2^n}$

$\lim_{n\to\infty} \frac{n^4}{2^n} = 0 \Rightarrow$ converges A.S.T.

$\sum_{n=1}^{\infty} \frac{n^4}{2^n}$

$\lim_{n\to\infty} \frac{(n+1)^4/2^{n+1}}{n^4/2^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^4 \cdot \frac{2^n}{2^{n+1}}$

$= 1\cdot \frac{1}{2} = \frac{1}{2} < 1$

Converges by ratio test

$\therefore$ Converges absolutely.

13. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$

$\lim_{n\to\infty} \frac{n}{n^2+1} = 0$

$f(x) = \frac{x}{x^2+1} \quad f'(x) = \frac{x^2+1-x(2x)}{(x^2+1)^2}$

$= \frac{-x^2+1}{(x^2+1)^2} \leq 0$

$\therefore$ converges by A.S.T.

$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \qquad b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n/(n^2+1)}{1/n}$

$= \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \Rightarrow$ diverges by LCT

$\therefore$ Converges conditionally.

14. $\sum_{n=1}^{\infty}\frac{\cos n\pi}{n} = (-1)^n\left(\frac{1}{n}\right)$

$\lim_{n\to\infty} \frac{1}{n} = 0$

$f(x) = \frac{1}{x} \quad f'(x) = -\frac{1}{x^2} < 0$

converges by A.S.T.

$\sum_{n=1}^{\infty} \frac{1}{n} \Rightarrow$ diverges by p-series test

$\therefore$ converges conditionally.

15. $\sum_{n=1}^{\infty}\frac{\cancel{\sin n}\ \cos n\pi}{n\sqrt{n}} = (-1)^n\left(\frac{1}{n^{3/2}}\right)$

$\frac{1}{(n+1)^{3/2}} < \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \frac{1}{n^{3/2}} = 0$

converges by A.S.T.

$\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by p-series test $(p>1)$

$\therefore$ Converges absolutely.

16. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n(n+1)}} > \frac{1}{\sqrt{(n+1)(n+2)}}$

$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+n}} = 0$

converges by A.S.T.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+n}} \qquad b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{1/\sqrt{n^2+n}}{1/n} = \lim_{n\to\infty} \sqrt{\frac{n^2}{n^2+n}} = 1$

Diverges by LCT

$\therefore$ Series converges conditionally.

17. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n+1}+\sqrt{n}} > \frac{1}{\sqrt{n+2}+\sqrt{n+1}}$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$$

converges by A.S.T.

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} \qquad b_n = \frac{1}{3\sqrt{n}}$$

$$\frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \frac{1}{3\sqrt{n}}$$

diverges by D.C.T.

∴ Series converges conditionally.

18. $\sum_{n=1}^{\infty}(-3)^{n+1}\frac{1}{n^2} = 3\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{3^n}{n^2}$

$$f(x) = \frac{3^x}{x^2}$$

$$f'(x) = \frac{x^2\cdot \ln 3\cdot 3^x - 2x\cdot 3^x}{x^4}$$

$$= \frac{x3^x(x\ln 3 - 2)}{x^4}$$

$$\lim_{n\to\infty} \frac{3^n}{n^2} = \infty$$

∴ Diverges by $n^{\text{th}}$ Term Test.

19. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{10n^{1.1}+1} > \frac{n+1}{10(n+1)^{1.1}+1}$

$$\lim_{n\to\infty} \frac{n}{10n^{1.1}+1}\cdot\frac{1/n}{1/n}$$

$$= \lim_{n\to\infty} \frac{1}{10n^{.1}+\frac{1}{n}} = 0$$

converges by A.S.T.

$$\sum_{n=1}^{\infty} \frac{n}{10n^{1.1}+1} \qquad b_n = \frac{1}{n^{.1}}$$

$$\lim_{n\to\infty} \frac{n/(10n^{1.1}+1)}{1/n^{.1}}$$

$$= \lim_{n\to\infty} \frac{n^{1.1}}{10n^{1.1}+1} = \frac{1}{10} \Rightarrow \text{Diverges}$$

∴ Series converges conditionally.

20. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n-1}{n}$

$$\lim_{n\to\infty} (-1)^{n+1}\left(\frac{n-1}{n}\right)$$

$$= \lim_{n\to\infty}(-1)^{n+1}\cdot\lim_{n\to\infty}\left(\frac{n-1}{n}\right)$$

$$= \lim_{n\to\infty}(-1)^{n+1}\cdot 1$$

$$= \lim_{n\to\infty}(-1)^{n+1} \Rightarrow DNE$$

∴ Series diverges, by $n^{\text{th}}$ Term Test.