9.4 Part II

Packet F

Use the Limit Comparison Test to determine convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{n}{n^2+2n+3} = a_n$

$b_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n/(n^2+2n+3)}{1/n}$

$= \lim_{n\to\infty} \frac{n^2}{n^2+2n+3} \cdot \frac{1/n^2}{1/n^2}$

$= \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}+\frac{3}{n^2}}$

$= 1$

By LCT, because $1/n$ diverges so does $\sum \frac{1}{n^2+2n+3}$ diverge.

2. $\sum_{n=1}^{\infty} \frac{3n+1}{n^3-4} = a_n$

$b_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(3n+1)/(n^3-4)}{1/n^2}$

$= \lim_{n\to\infty} \frac{3n^3+n^2}{n^3-4}$

$\overset{L}{=} \lim_{n\to\infty} \frac{9n^2+2n}{3n^2}$

$\overset{L}{=} \lim_{n\to\infty} \frac{18n+2}{6n}$

$\overset{L}{=} \lim_{n\to\infty} \frac{18}{6} = 3$

By LCT, because $1/n^2$ converges so do our series converge.

3. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n+1}} = a_n$

$b_n = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/n\sqrt{n+1}}{1/n^{3/2}}$

$= \lim_{n\to\infty} \frac{n^{3/2}}{n\sqrt{n+1}}$

$= \lim_{n\to\infty} \frac{n^{1/2}}{\sqrt{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$

$= 1$

By LCT, $\frac{1}{n^{3/2}}$ converges, so does Σ converge.

4. $\sum_{n=1}^{\infty} \frac{\sqrt{2n+1}}{n^2}$

$b_n = \frac{1}{n^{3/2}}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{2n+1}/n^2}{1/n^{3/2}}$

$= \lim_{n\to\infty} \frac{\sqrt{2n+1}}{\sqrt{n}}$

$= \lim_{n\to\infty} \sqrt{\frac{2n+1}{n}} = \lim_{n\to\infty} \sqrt{2+\frac{1}{n}}$

$= \sqrt{2}$

By LCT, since $\frac{1}{n^{3/2}}$ converges, so does our Σ.

Use the Ratio Test to determine convergence or divergence.

5. $\sum_{n=1}^{\infty} \frac{8^n}{n!}$ $\rho = \lim_{n\to\infty} \frac{8^{n+1}/(n+1)!}{8^n/n!}$ $= \lim_{n\to\infty} \frac{8^{n+1}}{8^n} \cdot \frac{n!}{(n+1)!}$ $= \lim_{n\to\infty} \frac{8}{n+1} = 0$ $\therefore \Sigma$ converges by the Ratio Test	6. $\sum_{n=1}^{\infty} \frac{5^n}{n^5}$ $\rho = \lim_{n\to\infty} \frac{5^{n+1}/(n+1)^5}{5^n/n^5}$ $= \lim_{n\to\infty} \frac{5^{n+1}}{5^n} \cdot \frac{n^5}{(n+1)^5}$ $= \lim_{n\to\infty} 5 \cdot \left(\frac{n}{n+1}\right)^5$ $= 5 \cdot 1 = 5$ $\therefore \Sigma$ diverges by the Ratio Test
7. $\sum_{n=1}^{\infty} \frac{n!}{n^{100}}$ $\rho = \lim_{n\to\infty} \frac{(n+1)!/(n+1)^{100}}{n!/n^{100}}$ $= \lim_{n\to\infty} \frac{(n+1)!}{n!} \cdot \left(\frac{n}{n+1}\right)^{100}$ $= \lim_{n\to\infty} (n+1) \cdot 1$ $= \infty$ $\therefore \Sigma$ diverges by the Ratio Test	8. $\sum_{n=1}^{\infty} \frac{n^3}{(2n)!}$ $\rho = \lim_{n\to\infty} \frac{(n+1)^3/[2(n+1)]!}{n^3/(2n)!}$ $= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 \cdot \frac{2n!}{(2n+2)!}$ $= \lim_{n\to\infty} 1 \cdot \frac{1}{(2n+1)(2n+2)} = 0$ $\therefore \Sigma$ converges by Ratio Test

Determine convergence or divergence for each of the series. Indicate the test you used.

9. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n+200}$

$$\lim_{n\to\infty} \frac{n}{n+200} = 1$$

$\therefore$ Σ diverges by nth Term Test.

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n+3}{n^2\sqrt{n}} = a_n$

$$b_n = \frac{1}{n^{3/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(n+3)/n^2\sqrt{n}}{1/n^{3/2}}$$

$$= \lim_{n\to\infty} \left(\frac{n+3}{n}\right) = 1$$

$\therefore$ Σ converges by LCT (because $\frac{1}{n^{3/2}}$ converges)

11. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n!}$

$$\rho = \lim_{n\to\infty} \frac{(n+1)^2/(n+1)!}{n^2/n!}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{n!}{(n+1)!}$$

$$= 1 \cdot 0 = 0$$

$\therefore$ Σ converges by Ratio Test

12. $\displaystyle\sum_{n=1}^{\infty} \frac{4n^3+3n}{n^5-4n^2+1} = a_n$

Σ behaves like $\frac{1}{n^2} = b_n$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(4n^3+3n)/(n^5-4n^2+1)}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{4n^5+3n^3}{n^5-4n^2+1} = 4$$

$\therefore$ Σ converges by LCT (because $1/n^2$ converges).

13. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{1/n(n+1)}{1/n^2}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right) = 1$$

$\therefore \sum$ converges by L.C.T.

14. $\frac{2}{1\cdot 3\cdot 4}+\frac{3}{2\cdot 4\cdot 5}+\frac{4}{3\cdot 5\cdot 6}+\frac{5}{4\cdot 6\cdot 7}+\cdots$

$$\sum_{n=1}^{\infty} \frac{(n+1)}{n(n+2)(n+3)}$$

$$b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{(n+1)/n(n+2)(n+3)}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{n(n+1)}{(n+2)(n+3)}$$

$$= \lim_{n\to\infty} \frac{n^2+n}{n^2+5n+6} = 1$$

$\therefore \sum$ converges by L.C.T.

15. $\frac{1}{3^1}+\frac{2}{3^2}+\frac{3}{3^3}+\frac{4}{3^4}+\cdots$

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

$$\rho = \lim_{n\to\infty} \frac{(n+1)/3^{n+1}}{n/3^n}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)\cdot \frac{3^n}{3^{n+1}}$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(\frac{n+1}{n}\right) = \frac{1}{3} < 1$$

$\therefore \Sigma$ converges by Ratio Test

16. $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{4\sqrt{4}}+\cdots$

$$\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$$

Converges by p-series Test.

17. $\sum_{n=1}^{\infty} \frac{1}{2+\sin^2 n}$

$\lim_{n\to\infty} \frac{1}{2+\sin^2 n} = DNE$

$\therefore$ Diverges by n^{th}-Term Test

$\sin^2 n$ oscillates between 0 and 1, thereby causing $\frac{1}{2+\sin^2 n} \geq \frac{1}{2}$ which is not 0.

18. $\sum_{n=1}^{\infty} \frac{4+\cos n}{n^3}$

$$\begin{array}{ccccc} -1 & \leq & \cos n & \leq & 1 \\ +4 & & +4 & & +4 \\ \hline 3 & \leq & \cos n + 4 & \leq & 5 \end{array}$$

$\therefore \frac{4+\cos n}{n^3} \leq \frac{5}{n^3} = 5 \cdot \frac{1}{n^3}$

since $\frac{1}{n^3}$ converges, $\frac{5}{n^3}$ converges

By Direct Comparison Test, the series converges.

19. $\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$

$\rho = \lim_{n\to\infty} \frac{(n+1)^{n+1}/(2(n+1))!}{n^n/(2n)!}$

$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{(2n)!}{(2n+2)!}$

$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} \cdot \frac{(n+1)}{1} \cdot \frac{1}{(2n+1)(2(n+1))}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \frac{1}{2(2n+1)}$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \cdot \frac{1}{2} \cdot \frac{1}{2n+1}$

$= \lim_{n\to\infty} \frac{e}{2} \cdot \frac{1}{2n+1} = 0$

$\therefore$ Converges by Ratio Test

20. $\sum_{n=1}^{\infty} \frac{4^n+n}{n!} = \sum_{n=1}^{\infty} \frac{4^n}{n!} + \sum_{n=1}^{\infty} \frac{n}{n!}$

$\rho = \lim_{n\to\infty} \frac{4^{n+1}/(n+1)!}{4^n/n!}$

$= \lim_{n\to\infty} \frac{4^{n+1}}{4^n} \cdot \frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} \frac{4}{n+1} = 0 < 1$ $\therefore$ converges by R.T.

$\rho = \lim_{n\to\infty} \frac{(n+1)/(n+1)!}{n/n!}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right) \cdot \frac{n!}{(n+1)!}$

$= \lim_{n\to\infty} \frac{1}{n} = 0 < 1$ $\therefore$ converges by R.T.

$\therefore \Sigma$ converges, sum of 2 convergent series