Answers 9.4 Part I

Packet E

1. $\sum_{k=1}^{\infty} \frac{1}{k+2}$

$$\int_1^{\infty} \frac{1}{x+2}\,dx = \lim_{b\to\infty} \ln(x+2)\Big|_1^{\infty}$$

$$= \infty$$

$\therefore$ Diverges by IT

2. $\sum_{k=1}^{\infty} \frac{k}{1+k^2}$

$$\int_1^{\infty} \frac{x}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \int_1^b \frac{x}{1+x^2}\,dx$$

$$= \lim_{b\to\infty} \left(\tfrac{1}{2}\ln(1+x^2)\right)\Big|_1^b$$

$$= \infty$$

$\therefore$ Diverges by IT

3. $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k+2}}$

$$\int_1^{\infty} (x+2)^{-1/2}\,dx$$

$$= \lim_{b\to\infty} \int_1^b (x+2)^{-1/2}\,dx$$

$$= \lim_{b\to\infty} 2(x+2)^{1/2}\Big|_1^b$$

$$= \infty$$

$\therefore$ Diverges by IT

4. $\sum_{k=1}^{\infty} \frac{2}{1+k^2}$

$$= 2\lim_{b\to\infty} \int_1^{\infty} \frac{dx}{1+x^2}$$

$$= 2\lim_{b\to\infty} \tan^{-1}x\Big|_1^b$$

$$= 2(\pi/2) - 2\left(\frac{\pi}{4}\right)$$

$$= \pi - \frac{\pi}{2} = \frac{\pi}{2}$$

$\therefore$ Converges by IT

5. $\sum_{k=1}^{\infty} \frac{1}{10k+3}$

$$= \lim_{b\to\infty} \int_1^b \frac{1}{10x+3}\,dx$$

$$= \lim_{b\to\infty} \frac{1}{10}\ln(10x+3)\Big|_1^b$$

$$= \infty$$

$\therefore$ Diverges by IT

6. $\sum_{k=1}^{\infty} \frac{k}{e^k} = ke^{-k}$

Let $u = x$, $du = dx$; $dv = e^{-x}dx$, $v = -e^{-x}$

$$= \lim_{b\to\infty} \int_1^b xe^{-x}\,dx$$

$$= -xe^{-x} + \int_1^b e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x}\Big|_1^b$$

$$= \lim_{b\to\infty} \left[\left(\frac{-b}{e^b} - \frac{1}{e^b}\right) - \left(\frac{-1}{e^1} - \frac{1}{e^1}\right)\right]$$

$$= 2/e$$

$\therefore$ converges by IT

7. $\sum_{k=1}^{\infty} \frac{1}{(4+3k)^{3/2}}$

$= \lim_{b\to\infty} \int_1^b (4+3x)^{-3/2}\,dx$

$= \lim_{b\to\infty} -\frac{2}{3}(4+3x)^{-1/2}\Big|_1^b$

$= \lim_{b\to\infty} \left[-\frac{2}{3}(4+3b)^{-1/2} + \frac{2}{3}(4+3)^{-1/2}\right]$

$= \frac{2}{3}(7)^{-1/2}$

$\therefore$ Converges by IT

8. $\sum_{k=1}^{\infty} \frac{k^2}{1+k^3}$

$= \lim_{b\to\infty} \int_1^{\infty} \frac{x^2}{1+x^3}\,dx$

$= \lim_{b\to\infty} \frac{1}{3}\ln(1+x^3)\Big|_1^b$

$= \infty$

$\therefore$ Diverges by IT

9. $\sum_{k=1}^{\infty} ke^{-k^2}$

$= \lim_{b\to\infty} \int_1^b xe^{-x^2}\,dx$

$= \lim_{b\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^b$

$= 0 + \frac{1}{2e}$

$\therefore$ Converges by IT

10. $\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^2}$

Let $u = \ln x$

$du = \frac{dx}{x}$

$= \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^2}\,dx$

$= \lim_{b\to\infty} \int_2^b u^{-2}\,du = \lim_{b\to\infty} \left(-u^{-1}\right)\Big|_2^b$

$= \lim_{b\to\infty} \left[-\frac{1}{\ln b} + \frac{1}{2}\right] = 1/2$

$\therefore$ Converges by IT

Use any test developed so far to decide about the convergence or divergence of the series. Give a reason for your conclusion.

11. $\sum_{k=1}^{\infty} \frac{k^2+1}{k^2+5}$

$\lim_{k\to\infty} \frac{k^2+1}{k^2+5} \overset{L}{=} \lim_{k\to\infty} \frac{2k}{2k} = 1 \neq 0$

$\therefore$ Diverges by "nth Term Test"

12. $\sum_{k=1}^{\infty} \left(\frac{3}{\pi}\right)^k$

$\frac{3}{\pi} < 1$

$\therefore$ Converges (geometric series)

13. $\displaystyle\sum_{k=1}^{\infty}\left[\left(\frac{1}{2}\right)^k + \frac{k-1}{2k+1}\right]$

$$\lim_{k\to\infty} \left(\tfrac{1}{2}\right)^k + \left(\frac{k-1}{2k+1}\right)$$

$$\overset{①}{=} \lim_{k\to\infty} \frac{1}{2} = \frac{1}{2} \neq 0$$

$\therefore$ Diverges by "n^{th} Term Test"

14. $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{k^2} + \frac{1}{2^k}\right)$

Converges by p-Series Test

Converges by Geometric Series $\left(\tfrac{1}{2} < 1\right)$

$\therefore$ Converges (sum of 2 convergent series)

15. $\displaystyle\sum_{k=1}^{\infty}\sin\left(\frac{k\pi}{2}\right)$

$$\lim_{k\to\infty} \sin\left(\frac{k\pi}{2}\right) \neq 0$$

$\therefore$ Diverges by "n^{th} Term Test"

16. $\displaystyle\sum_{k=1}^{\infty} k\sin\frac{1}{k}$

$$\lim_{k\to\infty} k \sin \frac{1}{k}$$

$$= \lim_{x\to 0} \frac{\sin x}{x} = 1 \neq 0$$

$\therefore$ Diverges by "n^{th} Term Test"

17. $\displaystyle\sum_{k=1}^{\infty} k^2 e^{-k}$

f	g
x^2	e^{-x}
$2x$	$-e^{-x}$
2	e^{-x}
0	$-e^{-x}$

$$= \lim_{b\to\infty} \int_1^{\infty} x^2 e^{-x}\,dx$$

$$= \lim_{b\to\infty} \left(-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}\right)\Big|_1^b$$

$$= \lim_{b\to\infty} \left\{\left(-b^2 e^{-b} - 2b e^{-b} - 2e^{-b}\right) - \left(-1^2 e^{-1} - 2(1) e^{-1} - 2e^{-1}\right)\right\}$$

$$= 5/e$$

$\therefore$ Converges by IT

18. $\displaystyle\sum_{k=2}^{\infty}\left(\frac{1}{k} - \frac{1}{k-1}\right)$

$$= \frac{1}{2} - 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{4} - \frac{1}{3} + \cdots$$

$$= -1 + \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{3} + \frac{1}{4} + \cdots$$

$$= -1$$

$\therefore$ Converges (collapsing series)

19. $\sum_{k=1}^{\infty} \frac{\tan^{-1} k}{1+k^2}$ Let $u = \tan^{-1} x$, $du = \frac{dx}{1+x^2}$ $= \lim_{b\to\infty} \int_1^b \frac{\tan^{-1} x}{1+x^2}\,dx$ $= \lim_{b\to\infty} \int_1^b u\,du = \lim_{b\to\infty} \frac{1}{2}u^2 \Big\|_1^b$ $= \lim_{b\to\infty} \frac{1}{2}(\tan^{-1} x)^2 \Big\|_1^b$ $= \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right] = \boxed{\frac{3\pi^2}{32}}$ $\therefore$ Converges by IT	20. $\sum_{k=2}^{\infty} \frac{1}{1+4k^2}$ $= \lim_{b\to\infty} \int_2^{\infty} \frac{1}{1+4x^2}\,dx$ $= \lim_{b\to\infty} \frac{1}{2}\tan^{-1}(2x) \Big\|_2^b$ $= \boxed{\frac{\pi}{4} - \frac{1}{2}\tan^{-1}(4)}$ $\therefore$ Converges by IT

21. For what values of p does $\sum_{k=3}^{\infty} 1/[n(\ln n)^p]$ converge? $p > 1$

Let $u = \ln x$
$du = \frac{dx}{x}$

$$\int_3^{\infty} \frac{dx}{x \cdot (\ln x)^p} = \int_3^{\infty} u^{-p}\,du$$

$$\begin{cases} = \dfrac{u^{-p+1}}{-p+1} & \text{if } p \neq 1 \\ = \ln u & \text{if } p = 1 \end{cases}$$

If $p > 1$
$\lim_{b\to\infty} \frac{\ln b^{-p+1}}{-p+1} \Big|_3^{\infty} = 0$
$\therefore$ Converges

If $p < 1$
$\lim_{b\to\infty} \frac{\ln b^{-p+1}}{-p+1} \Big|_3^{\infty} = -\infty$
$\therefore$ diverges

22. Does $\sum_{k=3}^{\infty} 1/[n \cdot \ln n \cdot \ln(\ln n)]$ converge or diverge?

$$\int_3^{\infty} \frac{dx}{x \ln x \ln(\ln x)}$$

$$= \lim_{b\to\infty} \ln(\ln(\ln x)) \Big|_3^b = \infty \quad \text{(but very slowly)}$$

$\boxed{\therefore \text{ Diverges by IT}}$