Math 604 – AP Calculus BC Name KEY

Packet D: Infinite Series (corresponds to Section 9.1)

I. Geometric Series. A series of the form $\sum_{k=1}^{\infty} ar^{k-1} = a + ar + ar^2 + ar^3 + \cdots$ where $a \neq 0$ is called a **geometric series.** This series converges with sum $S = a/(1-r)$ if $|r| < 1$, but diverges if $|r| \geq 1$.

Indicate whether the given series converges or diverges. If it converges, find its sum. *Hint:* it may help you to write out the first few terms of the series.

1. $\sum_{k=1}^{\infty} \left(\frac{1}{5}\right)^k$

Geometric series where $\frac{1}{5} < 1$

$S = \frac{1/5}{1 - 1/5} = \frac{1/5}{4/5} = \frac{1}{4}$

$\therefore$ Series converges with a sum of $\frac{1}{4}$.

2. $\sum_{k=0}^{\infty} \left[2\left(\frac{1}{3}\right)^k + 3\left(\frac{1}{6}\right)^k\right]$

$= \sum_{k=0}^{\infty} 2\left(\frac{1}{3}\right)^k + \sum_{k=0}^{\infty} 3\left(\frac{1}{6}\right)^k$

Both are geometric series where $r < 1$.

$S = \frac{2}{1 - \frac{1}{3}} = 3 \quad \Big\} \quad S = \frac{3}{1 - \frac{1}{6}} \quad \frac{18}{5}$

$\therefore$ Series converges to a sum of $\frac{33}{5}$.

3. $\sum_{k=1}^{\infty} \frac{k-3}{k}$

$= \sum_{k=1}^{\infty} 1 - \frac{3}{k}$

As k approach infinity, $\frac{3}{k} \to 0$, and the series is essential an infinity sum of 1's $\Rightarrow$ Diverges.

4. $\sum_{k=1}^{\infty} \left(\frac{4}{3}\right)^k$

A geometric series where $\frac{4}{3} > 1$

$\therefore$ Diverges.

5. $\sum_{k=1}^{\infty} \frac{k!}{10^k}$

When k is less than 25, $k! < 10^k$. But when $k \geq 25$, $k! > 10^k$.

Therefore, the series diverges because the terms are increasing for $k \geq 25$.

6. $\sum_{k=3}^{\infty} \left(\frac{e}{\pi}\right)^{k-1}$

Geometric Series where $\frac{e}{\pi} < 1$.

$$S = \frac{e^2/\pi^2}{1 - e/\pi} \cdot \frac{\pi^2}{\pi^2}$$

$$= \frac{e^2}{\pi^2 - e\pi}$$

$\therefore$ Series converges with a sum of $e^2/(\pi^2 - e\pi) \approx 5.55622$

7. $\sum_{k=1}^{\infty} \left(\frac{3}{(k+1)^2} - \frac{3}{k^2}\right)$

$$= \left(\frac{3}{2^2} - \frac{3}{1^2}\right) + \left(\frac{3}{3^2} - \frac{3}{2^2}\right) + \left(\frac{3}{4^2} - \frac{3}{3^2}\right) + \ldots$$

Re-associating, we get

$$= -3 + \left(\frac{3}{2^2} - \frac{3}{2^2}\right) + \left(\frac{3}{3^2} - \frac{3}{3^2}\right) + \ldots$$

$$= -3 + 0 + 0 + \ldots$$

$$= -3$$

$\therefore$ Series converges to -3.

8. $\sum_{k=4}^{\infty} \frac{4}{k-3}$

We know the harmonic series $\sum \frac{1}{k}$ diverges and $\frac{1}{k} < \frac{4}{k-3}$, therefore the series diverges, also.

COLLAPSING SERIES. A geometric series is one of the few series where we can actually calculate S_n; a collapsing series is another.

Example. Show that the following series converges and find its sum.

$$\sum_{k=1}^{\infty}\left(\frac{1}{(k+2)(k+3)}\right)$$

Solution. Use a partial fraction decomposition to write

$$\frac{1}{(k+2)(k+3)}=\frac{1}{k+2}-\frac{1}{k+3}$$

Then,

$$S_n=\sum_{k=1}^{n}\left(\frac{1}{k+2}-\frac{1}{k+3}\right)=\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots+\left(\frac{1}{n+2}-\frac{1}{n+3}\right)$$

$$=\frac{1}{3}-\frac{1}{n+3}$$

Therefore,

$$\lim_{n\to\infty}S_n=\frac{1}{3}$$

Use the above example to determine whether the given series converges or diverges. If it converges, find its sum.

9. $\sum_{k=2}^{\infty}\left(\frac{2}{k(k-1)}\right) = \sum_{k=2}^{\infty}\left(\frac{2}{k-1}-\frac{2}{k}\right)$

$=\left(\frac{2}{1}-\frac{2}{2}\right)+\left(\frac{2}{2}-\frac{2}{3}\right)+\left(\frac{2}{3}-\frac{2}{4}\right)+\cdots$

$=\frac{2}{1}+\left(\frac{2}{2}-\frac{2}{2}\right)+\left(\frac{2}{3}-\frac{2}{3}\right)+\cdots$

$=2$

$\therefore$ Series converges to 2.

10. $\sum_{k=1}^{\infty}\left(\frac{4}{(4k-3)(4k+1)}\right)$

$=\sum_{k=1}^{\infty}\left(\frac{1}{4k-3}-\frac{1}{4k+1}\right)$

$=\left(\frac{1}{1}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{13}\right)+\cdots$

$=1+\left(\frac{1}{5}-\frac{1}{5}\right)+\left(\frac{1}{9}-\frac{1}{9}\right)+\cdots$

$=1$

$\therefore$ Series converges to 1.

11. $\sum_{k=1}^{\infty}\left(\frac{6}{(2k-1)(2k+1)}\right)$

$= \sum_{k=1}^{\infty}\left(\frac{3}{2k-1} - \frac{3}{2k+1}\right) = \left(3 - \frac{3}{2}\right)$

$+ \left(\frac{3}{2} - \frac{3}{3}\right) + \left(\frac{3}{3} - \frac{3}{5}\right) + \left(\frac{3}{5} - \frac{3}{7}\right) + \ldots$

$= 3$ [redacted] $+ \left(\frac{3}{3} - \frac{3}{3}\right) + \left(\frac{3}{5} - \frac{3}{5}\right) + \ldots$

$=$ [redacted] $3 \rightarrow \therefore$ Series converges to 3 [redacted]

12. $\sum_{k=1}^{\infty}\left(\frac{40k}{(2k-1)^2(2k+1)^2}\right)$

$= \sum_{k=1}^{\infty}\left(\frac{5}{(2k-1)^2} - \frac{5}{(2k+1)^2}\right)$

$= \left(\frac{5}{1^2} - \frac{5}{3^2}\right) + \left(\frac{5}{3^2} - \frac{5}{5^2}\right) + \ldots$

$= 5$

$\therefore$ Series converges to 5.

13. $\sum_{k=1}^{\infty}\left(\frac{2k+1}{k^2(k+1)^2}\right)$

$= \sum_{k=1}^{\infty}\left(\frac{1}{k^2} - \frac{1}{(k+1)^2}\right)$

$= \left(\frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{1}{2^2} - \frac{1}{3^2}\right) + \ldots$

$= 1$

$\therefore$ Converges to 1.

14. $\sum_{k=1}^{\infty}\left(\frac{1}{\ln(k+2)} - \frac{1}{\ln(k+1)}\right)$

$= \left(\frac{1}{\ln 3} - \frac{1}{\ln 2}\right) + \left(\frac{1}{\ln 4} - \frac{1}{\ln 3}\right) + \ldots$

$= -\frac{1}{\ln 2}$

$\therefore$ Series converges to $-(\ln 2)^{-1}$, or around -1.4427.

15. $\sum_{k=1}^{\infty}\left(\tan^{-1}(k) - \tan^{-1}(k+1)\right)$

$= \tan^{-1}(1) - \tan^{-1}(2) + \tan^{-1}(2) - \ldots$

$= \tan^{-1}(1)$

$= \pi/4$

$\therefore$ Series converges to $\pi/4$.

16. $\sum_{k=1}^{\infty}\left(\frac{2^{2k}}{(2k)!} - \frac{2^{2k+1}}{(2k+1)!}\right)$

$= \frac{(-2)^2}{2!} + \frac{(-2)^3}{3!} + \frac{(-2)^4}{4!} + \frac{(-2)^5}{5!} + \frac{(-2)^6}{6!} + \ldots$

We know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

so if $x = -2$ we get

$e^{-2} = 1 - 2 + \frac{2^2}{2!} - \frac{2^3}{3!} + \frac{2^4}{4!} - \frac{2^5}{5!} + \frac{2^6}{6!} \ldots$

Add 1 we get:

$\boxed{e^{-2} + 1}$