Math 604 – AP Calculus BC Name KEY

9.2 & 9.3 Taylor & Maclaurin Polynomial Practice – Part C

Maclaurin Series at $x=0$: $f(x) \approx f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$

Use the table of Maclaurin series. Construct the first three nonzero terms and the general term of the Maclaurin series generated by the function, and give the interval of convergence.

1. $f(x)=\sin 2x$ substitute $2x$ for x in Maclaurin Series for $\sin x$

$$\sin 2x = 2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots + (-1)^n\frac{(2x)^{2n+1}}{(2n+1)!} + \ldots$$

$$= 2x - \frac{4x^3}{3} + \frac{4x^5}{15} - \ldots + (-1)^n\frac{(2x)^{2n+1}}{(2n+1)!} + \ldots$$

(converges for all real x)

2. $f(x)=\tan^{-1}x^2$ substitute x^2 for x in Maclaurin series for $\tan^{-1}x$

$$\tan^{-1}x^2 = x^2 - \frac{(x^2)^3}{3} + \frac{(x^2)^5}{5} - \ldots + (-1)^n\frac{(x^2)^{2n+1}}{2n+1} + \ldots$$

$$= x^2 - \frac{x^6}{3} + \frac{x^{10}}{5} - \ldots + (-1)^n\frac{x^{4n+2}}{2n+1} + \ldots$$

(for $|x^2| \le 1 \Rightarrow [-1,1]$)

3. $f(x)=e^{-2x}$ substitute $-2x$ for x in Maclaurin Series for e^x

$$e^{-2x} = 1 + (-2x) + \frac{(-2x)^2}{2!} + \ldots + \frac{(-2x)^n}{n!} + \ldots$$

$$= 1 - 2x + 2x^2 - \ldots + \frac{(-1)^n 2^n x^n}{n!} + \ldots$$

(converges for all real x)

Taylor Series at $x = a$:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Find the Taylor Polynomials of order 3 centered at the given points.

4. $f(x) = \dfrac{1}{x+1}$; $x = 2$

$f'(x) = -(x+1)^{-2}$

$f''(x) = 2(x+1)^{-3}$

$f'''(x) = -6(x+1)^{-4}$

$$P_3(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^2}{3!}$$

$$= \frac{1}{3} - \frac{1}{9}(x-2) + \frac{2}{27}\frac{(x-2)^2}{2} - \frac{6}{81}\frac{(x-2)^3}{6}$$

$$= \frac{1}{3} - \frac{(x-2)}{9} + \frac{(x-2)^2}{27} - \frac{(x-2)^3}{81}$$

5. $f(x) = e^{x/2}$; $x = 1$

$f'(x) = \frac{1}{2}e^{x/2}$

$f''(x) = \frac{1}{4}e^{x/2}$

$f'''(x) = \frac{1}{8}e^{x/2}$

$$P_3(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^2}{3!}$$

$$= e^{1/2} + \frac{1}{2}e^{1/2}(x-1) + \frac{1}{4}e^{1/2}\frac{(x-1)^2}{2!} + \frac{1}{8}e^{1/2}\frac{(x-1)^3}{3!}$$

$$= e^{1/2} + e^{1/2}\frac{(x-1)}{2^1} + e^{1/2}\frac{(x-1)^2}{2^2\,2!} + e^{1/2}\frac{(x-1)^3}{2^3\,3!}$$

6. Let f be a function that has derivatives of all orders for all real numbers. Assume that

$f(1) = 4$
$f'(1) = -1$
$f''(1) = 3$
$f'''(1) = 2$

a) Write the third order Taylor polynomial for f at $x = 1$ and use it to approximate $f(1.2)$.

$$P_3(x) = f(a) + f'(a)(x-1) + f''(a)\frac{(x-1)^2}{2!} + f'''(a)\frac{(x-1)^3}{3!}$$

$$P_3(x) = 4 - (x-1) + \frac{3}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

$$P_3(1.2) = 4 - (1.2-1) + \frac{3}{2}(1.2-1)^2 + \frac{1}{3}(1.2-1)^3$$

$$\approx \boxed{3.863}$$

b) Write the second order Taylor polynomial for f', the derivative of f, at $x = 1$ and use it to approximate $f'(1.2)$.

Since the Taylor series of $f'(x)$ can be obtained by differentiating terms of the Taylor Series of $f(x)$, the second order Taylor polynomial of $f'(x)$ is given by:

$$P_3'(x) = -1 + 3(x-1) + (x-1)^2$$

$$P_3'(1.2) = -1 + 3(1.2-1) + (1.2-1)^2$$

$$\approx \boxed{-.36}$$

Taylor Formula with Remainder at $x = a$:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where the remainder (error) is $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ and c is some point between x and a.

Find a good estimate for the maximum value of the given expression, c, being in the given interval. Answers may vary depending upon the technique used.

7. $|\tan x + \sec x|$; $[-\frac{\pi}{4}, 0]$

$\leq |\tan c| + |\sec c|$

$\leq |-1| + |\sqrt{2}|$

$\leq \boxed{2.5}$

8. $\left|\frac{4}{c+2}\right|$; $[-1.5, 2]$

$\leq \frac{4}{|-1.5+2|}$

$\leq \frac{4}{.5} \leq \boxed{8}$

9. $\left|\frac{\sin c}{c+1}\right|$; $[0, 1]$

$\leq \frac{|\sin c|}{|c+1|}$

$\leq \frac{\sin 1}{1} \leq \boxed{.842}$

10. $\left|\frac{c^4 - c}{\sin c}\right|$; $[1, 3]$

$\leq \frac{|c^4| + |c|}{|\sin c|}$

$\leq \frac{3^4 + 3}{\sin 3} \leq \boxed{596}$

For 11 and 12, find a formula for $R_6(x)$, the remainder for the Taylor polynomial of order 6 based at a. Then estimate $|R_6(0.5)|$, that is, give a good upper bound for it.

11. e^{-x}; $a = 0$	12. $\frac{1}{x-2}$; $a = 1$
$f' = -e^{-x}$ $f'' = e^{-x}$ $\vdots$ $f^{(7)} = -e^{-x}$ $R_6(x) = \frac{-e^{-c}(x-0)^7}{7!}$ $\lvert R_6(0.5)\rvert \le \left\lvert \frac{e^0(.5)^7}{7!} \right\rvert$ $\le \boxed{1.5501 \times 10^{-6}}$ where $R_6(x) = \left\lvert \frac{x^7}{7!} \right\rvert$	$f' = -(x-2)^{-2}$ $f'' = 2(x-2)^{-3}$ $f''' = -6(x-2)^{-4}$ $f^{(4)} = 24(x-2)^{-5}$ $\vdots$ $f^{(7)} = 7!(x-2)^{-8}$ $R_6(x) = \frac{7!(c-2)^{-8}}{7!}(x-1)^7$ $\lvert R_6(.5)\rvert \le \left\lvert \frac{(1-2)^{-8}}{1} \cdot (.5-1)^7 \right\rvert$ $\le \boxed{0.0078125}$ where $R_6(x) = \lvert (x-1)^7 \rvert$

13. Find the third order Maclaurin polynomial for $(1+x)^{3/2}$ and estimate the error $R_3(x)$ if $-0.1 \le x \le 0$.

$f' = \frac{3}{2}(1+x)^{1/2}$ $\qquad f'''' = \frac{9}{16}(1+x)^{-5/2}$

$f'' = \frac{3}{4}(1+x)^{-1/2}$

$f''' = -\frac{3}{8}(1+x)^{-3/2}$

$P_3(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!}$

$P_3(x) = 1 + \frac{3}{2}x + \frac{3}{8}x^2 - \frac{1}{16}x^3$

$|R_3(x)| = \left|\frac{9}{16}(1+c)^{-5/2}\frac{x^4}{4!}\right| \le \left|\frac{9}{16}(.9)^{-5/2} \cdot \frac{(-.1)^4}{4!}\right| \le \boxed{3.051 \times 10^{-6}}$

14. Note that the fourth-order Maclaurin polynomial for $\sin x$ is really of third degree since the coefficient of x^4 is 0. Thus

$$\sin x = x - \frac{x^3}{6} + R_4(x)$$

I

Show that if $0 \le x \le 0.5$, $|R_4(x)| \le 0.0002605$. Use this result to approximate $\int_0^{0.5} \sin x \, dx$ and give an estimate of the error.

I

$f' = \cos x \qquad f''' = -\cos x \qquad f^{(v)} = \cos x$

$f'' = -\sin x \qquad f^{(iv)} = \sin x$

$|R_4(x)| = \left|\cos c \cdot \frac{x^5}{5!}\right| \le \left|1 \cdot \frac{.5^5}{5!}\right| \le 2.605 \times 10^{-4}$ ✓

II

$\int_0^{0.5} \sin x \, dx = \int_0^{0.5} \left(x - \frac{x^3}{6}\right) dx$

$= \frac{x^2}{2} - \frac{x^4}{24}\Big|_0^{.5} = \frac{.5^2}{2} - \frac{.5^4}{24} \approx \boxed{.122396}$

Est. of error $\int_0^{.5} (2.605 \times 10^{-4}) \, dx$

$= 2.605 \times 10^{-4} \Big|_0^{.5} \approx \boxed{1.31 \times 10^{-4}}$