Math 604 – AP Calculus BC Name: KEY

9.2 Taylor & Maclaurin Polynomials – Part A

Maclaurin Series at $x=0$: $f(x) \approx f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots+\frac{f^{(n)}(0)}{n!}x^n+\ldots=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}x^k$

Find the Maclaurin Polynomials of order 4 for $f(x)$ and use it to approximate $f(0.25)$.

1. $f(x)=e^{2x}$

$f' = 2e^{2x}$
$f'' = 4e^{2x}$
$f''' = 8e^{2x}$
$f^{(4)} = 16e^{2x}$

$P_4(x) = 1 + 2x + \frac{4x^2}{2!} + \frac{8x^3}{3!} + \frac{16x^4}{4!}$

$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4$

$P_4(.25) = 1.6484375$

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2. $f(x)=\sin 2x$

$f' = 2\cos 2x$
$f'' = -4\sin 2x$
$f''' = -8\cos 2x$
$f^{(4)} = 16\sin x$

$P_4(x) = 0 + 2x - 0 - \frac{8x^3}{3!} + 0$

$= 2x - \frac{4}{3}x^3$

$P_4(.25) = .4791667$

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3. $f(x)=\ln(x+1)$

$f' = (x+1)^{-1}$
$f'' = -(x+1)^{-2}$
$f''' = 2(x+1)^{-3}$
$f^{(4)} = -6(x+1)^{-4}$

$P_4(x) = 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!}$

$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4$

$P_4(.25) = .2229817708$

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4. $f(x)=\tan^{-1} x$

$f' = (1+x^2)^{-1}$

$f'' = -2x(1+x^2)^{-2}$

$f''' = (6x^2-2)(1+x^2)^{-3}$

$f^{(4)} = (-24x^3+24x)(1+x^2)^{-4}$

$P_4(x) = 0 + x - 0 - \frac{2x^3}{3!} + 0$

$= x - \frac{1}{3}x^3$

$P(.25) = .2447916\overline{7}$

Taylor Series at $x=a$:

$$f(x)\approx f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\ldots+\frac{f^{(n)}(a)}{n!}(x-a)^n+\ldots=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

Find the Taylor Polynomials of order 3 centered at the given points.

5. $f(x)=e^x$; $x=2$

$f'=e^x$

$f''=e^x$

$f'''=e^x$

$$P_3(x)=e^2+e^2(x-2)+e^x\frac{(x-2)^2}{2!}+e^x\frac{(x-2)^3}{3!}$$

$$P_3(x)=e^2\left(1+(x-2)+\tfrac{1}{2}(x-2)^2+\tfrac{1}{6}(x-2)^3\right)$$

6. $f(x)=\tan x$; $x=\dfrac{\pi}{4}$

$f'=\sec^2 x$

$f''=2\sec^2 x\tan x$

$f'''=2\sec^2 x(\sec^2 x+2\tan^2 x)$

$$P_3(x)=1+2\left(x-\tfrac{\pi}{4}\right)+\frac{4\left(x-\frac{\pi}{4}\right)^2}{2!}+\frac{16\left(x-\frac{\pi}{4}\right)^3}{3!}$$

$$P_3(x)=1+2\left(x-\tfrac{\pi}{4}\right)+2\left(x-\tfrac{\pi}{4}\right)^2+\tfrac{8}{3}\left(x-\tfrac{\pi}{4}\right)^3$$

7. $f(x)=\tan^{-1}x$; $x=1$

$f'=(1+x^2)^{-1}$

$f''=-2x(1+x^2)^{-2}$

$f'''=(6x^2-2)(1+x^2)^{-3}$

$$P_3(x)=\frac{\pi}{4}+\frac{1}{2}(x-1)-\frac{1}{2}\frac{(x-1)^2}{2!}+\frac{1}{2}\frac{(x-1)^3}{3!}$$

$$P_3(x)=\frac{\pi}{4}+\frac{1}{2}(x-1)-\frac{1}{4}(x-1)^2+\frac{1}{12}(x-1)^3$$

8. Find the Taylor Polynomial of order 3 centered at 2 for $f(x) = x^3 - 2x^2 + 3x + 5$ and show that it is an exact representation of $f(x)$.

$f' = 3x^2 - 4x + 3$ $\quad f'(2) = 12 - 8 + 3 = 7$

$f'' = 6x - 4$ $\quad f''(2) = 12 - 4 = 8$

$f''' = 6$ $\quad f'''(2) = 6$

$P_3(x) = f(2) + f'(2)(x-2) + f''(2)\frac{(x-2)^2}{2!} + f'''(2)\frac{(x-2)^3}{3!}$

$* \ P_3(x) = 11 + 7(x-2) + 4(x-2)^2 + (x-2)^3$

$= 11 + 7x - 14 + 4(x^2 - 4x + 4) + x^3 - 6x^2 + 12x - 8$

$= x^3 - 2x^2 + 3x + 5 = f(x)$

9. Find the Maclaurin Polynomial of order n for $f(x) = 1/(1-x)$. Then use it with $n = 4$ to approximate each of the following.

(a) $f(0.1)$ (b) $f(0.5)$ (c) $f(0.9)$ (d) $f(2)$

$f' = (1-x)^{-2}$

$f'' = 2(1-x)^{-3}$

$f''' = 6(1-x)^{-4}$

$\vdots$

$f^{(n)} = n!(1-x)^{-n-1}$

a) 1.1111

b) 1.9375

c) 4.0951

d) 31

$\therefore (1-x)^{-1} \approx 1 + x + x^2 + x^3 + x^4 + \dots + x^n$

How does this example show you that the Maclaurin series can be exceedingly poor if x is far from zero?

When $x = 2$, $f(2) = \frac{1}{1-2} = -1$ but the approximation is 31! The estimate is exceedingly the farther x is from 0.

[Error is $\left| (n+1)!(1-c)^{-n-2} \cdot \frac{x^{n+1}}{(n+1)!} \right|$ or $\left| (1-c)^{-n-2} x^{n+1} \right|$]